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Classical solutions of two- and four-dimensional σ-models interpolating between instanton and meron configurations

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Abstract. We obtain an infinite class of exact time-dependent solutions of the classical two- and four-dimensional σ-models. In Euclidean space these solutions interpolate continuously between the corresponding meron and instanton solutions.

1. Introduction

Recently the σ-model in both two and four dimensions has received a lot of attention (Schroer 1977, Gava and Jengo 1978, de Alfaro *et al* 1978, Ferrara *et al* 1978), the main motivation being the similarities—like asymptotic freedom, conformal invariance, etc—between two-dimensional σ-model and four-dimensional Yang–Mills (YM) theories (Belavin and Polyakov 1975, Migdal 1976). Another motivation is that the two-dimensional σ-model is equivalent to the two-dimensional Heisenberg ferromagnet. For example, Gross (1978) and, particularly, de Alfaro *et al* (1978) have discussed both meron and instanton solutions of a Euclidean O(3) σ-model in two dimensions. The latter authors have also generalised these results and have obtained instanton and meron solutions of the four-dimensional nonlinear O(5) σ-model in the case when the Lagrangian has fourth-order derivative terms. It is of course well known that these theories are marred by the problem of ghost states, but a recent paper (Narnhofer and Thirring 1978) indicates that under special conditions observable quantities may not be affected by ghosts.

At this stage it is worthwhile to point out that for four-dimensional YM theories one can obtain a class of solutions which continuously interpolate between the instanton and meron solutions (Schechter 1977, Luscher 1977, Cervero *et al* 1977, Wada 1978). It is then natural to enquire if one can derive similar solutions for the σ-model. The answer to this question is not very obvious, because the two models differ in an important respect, i.e. the lack of multiple or θ-vacuum in the σ-model (Bitter *et al* 1979). The purpose of this paper is to exhibit a class of solutions of both two- and four-dimensional σ-models which interpolate continuously between the instanton and meron solutions found previously (de Alfaro *et al* 1978).

The plan of the paper is as follows. In § 2 we obtain the general solutions of the two-dimensional O(3) σ-model and discuss their symmetry properties. In § 3 we do a similar job for the four-dimensional O(5) σ-model. Since there may be a problem of ghost states with this Lagrangian, it may be worthwhile to enquire if the standard

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four-dimensional σ-model possesses instanton and meron solutions. In § 4 we show that this is possible provided a quartic self-interaction term is added to the O(4) σ-model.

2. Solutions of two-dimensional σ-model

Following de Alfaro *et al* (1978) we start with the Lagrangian density (with Euclidean metric)

$$\mathscr{L} = \tfrac{1}{2}\partial_\mu\phi_\alpha\,\partial_\mu\phi_\alpha - \tfrac{1}{2}m(x)(\phi_\alpha^2 - 1), \tag{2.1}$$

where $\phi_\alpha(x_1, x_2)$ is a three-component unit vector and $m(x)$ is a multiplier field. The equations of motion which follow from here are

$$(\Box + m(x))\phi_\alpha = 0, \tag{2.2a}$$

$$\phi_\alpha^2 = 1. \tag{2.2b}$$

Using (2.2a) and (2.2b) one has

$$\Box\phi_\alpha = (\phi_\beta \Box \phi_\beta)\phi_\alpha. \tag{2.3}$$

General solutions. Remembering that equation (2.3) is conformal invariant, we start with the ansatz

$$\begin{aligned} \phi_\alpha &= [x_\mu/(x^2)^{1/2}]f(x^2)\delta_{\mu\alpha}, \qquad \alpha = 1, 2, \\ \phi_3 &= (1-f^2)^{1/2}, \end{aligned} \tag{2.4}$$

f being some arbitrary function of x^2. Substituting this ansatz in equation (2.3) we find that $f(x^2)$ must satisfy the equation

$$4x^4f'' + 4x^2f' + 4x^2ff'^2/(1-f^2) = f(1-f^2), \tag{2.5}$$

where $f' \equiv \mathrm{d}f(x^2)/\mathrm{d}x^2$. On using $x^2 = \mathrm{e}^t$ this equation reduces to

$$\frac{\mathrm{d}^2f}{\mathrm{d}t^2} + \frac{f}{1-f^2}\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)^2 - \frac{f}{4}(1-f^2) = 0, \tag{2.6}$$

whose solution is known to be (Murphy 1960)

$$f(x^2) = \mathrm{dn}(\log(x^2)^{1/2}|k), \tag{2.7}$$

where $\mathrm{dn}(y\,|\,k)$ is a Jacobi elliptic function.

Let us now show that this solution contains both instanton and meron configurations. First notice that, in the limit $k \to 0$, $f(x^2)$ goes to 1, so that

$$\begin{aligned} \phi_\alpha &= [x_\mu/(x^2)^{1/2}]\delta_{\mu\alpha}, \qquad \alpha = 1, 2, \\ \phi_3 &= 0, \end{aligned} \tag{2.8}$$

which is the meron solution of de Alfaro *et al* (1978). On the other hand, as $k \to 1$ we obtain

$$f(x^2) = \mathrm{sech}[\log(x^2)^{1/2}] = 2(x^2)^{1/2}/(1+x^2),$$

so that

$$\phi_\alpha = [2x_\mu/(1+x^2)]\delta_{\mu\alpha}, \qquad \alpha = 1, 2, \tag{2.9}$$
$$\phi_3 = (1-x^2)/(1+x^2),$$

which is the σ-model instanton solution. It must be pointed out here that anti-solutions can be obtained from the above solutions simply by changing $\phi_2 \rightarrow -\phi_2$, the only consequence being that they will have opposite topological charge densities compared with the usual ones. The infinite class of intermediate solutions $f(x^2)$ for $0<k<1$ oscillate around the meron configuration ($f=1$). The physical meaning of these configurations is not quite clear. It is likely that, as for the meron solution, they may also be related to quark confinement.

Symmetry properties of the general solution. As has been shown by de Alfaro *et al* (1978), the meron solution (2.8) is invariant under an O(2) ⊗ O(2) group, where one O(2) group is generated by the two-dimensional dilatation generator D($D \equiv ix\cdot\partial$), while the other corresponds to the complete (space plus internal) two-dimensional rotation. On the other hand, the instanton solution (2.9) is invariant under the O(3) group generated by complete three-dimensional rotations. It is then natural to enquire about the symmetry properties of the general solutions (2.7). Using $D \equiv ix\cdot\partial$ it is easy to see that, for the solution (2.7),

$$D\phi_1 = [-ix_1k/(x^2)^{1/2}]\,\mathrm{sn}\,\mathrm{cn}, \qquad D\phi_\alpha \neq 0. \tag{2.10}$$

On the other hand, for the general solutions (2.7),

$$(M_{12}+\Sigma_{12})\phi_\alpha = 0, \tag{2.11}$$

where

$$M_{12} = i(x_1\,\partial_2 - x_2\,\partial_1),$$
$$\Sigma_{\alpha\beta}\phi_\gamma = i(\delta_{\alpha\gamma}\phi_\beta - \delta_{\beta\gamma}\phi_\alpha). \tag{2.12}$$

In other words, the general solutions (2.7) are invariant under a complete two-dimensional rotation group O(2).

Energy–momentum tensor. Using the standard definition of traceless, divergenceless $\theta_{\mu\nu}$ given by

$$\theta_{\mu\nu} = -\tfrac{1}{2}\delta_{\mu\nu}(\partial_\lambda\phi_\alpha)^2 + \partial_\mu\phi_\alpha\,\partial_\nu\phi_\alpha, \tag{2.13}$$

it is straightforward to show that, for the general solutions (2.7),

$$\theta_{\mu\nu}(x\,|\,k) = (1-k)\theta_{\mu\nu}^{\mathrm{meron}}, \tag{2.14}$$

where

$$\theta_{\mu\nu}^{\mathrm{meron}} = (1/2x^2)(\delta_{\mu\nu} - 2x_\mu x_\nu/x^2). \tag{2.15}$$

It is amusing to note that a similar relation also exists for general solutions of four-dimensional YM theories. It is striking that the $\theta_{\mu\nu}$ for different values of k are simply related to each other. From here it is also clear that all these solutions have zero momentum. It also follows from (2.14) that $\theta_{\mu\nu} = 0$ for instanton solution (2.9).

Action. Using solution (2.7) in equation (2.1), it follows that

$$\mathcal{L} = (1/2x^2)[2\,\mathrm{dn}^2(\log(x^2)^{1/2}\,|\,k) - (1-k)]. \tag{2.16}$$

Needless to say that in the limit $k \to 0(1)$ we obtain the meron (instanton) Lagrangian density. These configurations do not lead to well-defined Euclidean action, but, on the other hand, one can obtain finite Minkowski action and energy by means of a suitable conformal transformation, as has been done by de Alfaro *et al* (1978) for the meron configuration.

Topological charge density. The topological charge density is defined as the Jacobian of the mapping $\phi_\alpha(x)$ of the space–time on the unit sphere:

$$D(x) = \frac{1}{8\pi} \epsilon_{\alpha\beta\gamma}\epsilon_{\mu\nu}\, \partial_\mu \phi_\alpha\, \partial_\nu \phi_\beta \phi_\gamma. \tag{2.17}$$

For the general solution (2.7) we find that

$$D(x) = (k/4\pi x^2)\text{cn dn}. \tag{2.18}$$

As expected, the topological charge $Q = \int D(x)\, d^2x$ is indefinite for these solutions. As $k \to 1$, however, we get back $Q = 1$.

3. Four-dimensional σ-model

The Lagrangian (with Euclidean metric) that we will consider is given by (de Alfaro 1978)

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu\, \partial_\nu \phi_\alpha)^2 + \tfrac{1}{2}m(x)(\phi_\alpha^2 - 1), \qquad \alpha = 1, \ldots, 5. \tag{3.1}$$

The reason for considering such a higher-derivative Lagrangian is our insistence that ϕ_α be a dimensionless field so that the theory is conformal invariant. After eliminating the multiplier field $m(x)$, this leads to the equations of motion

$$(\Box)^2\phi_\alpha = [\phi_\beta(\Box)^2\phi_\beta]\phi_\alpha. \tag{3.2}$$

A nice thing about this four-dimensional model is that the interesting features of the two-dimensional σ-model—like asymptotic freedom and dynamical generation of mass—are also present.

Proceeding as in the last section, general solutions of (3.2) can be shown to be

$$\begin{aligned} \phi_\alpha &= [x_\mu/(x^2)^{1/2}]\, \text{dn}(\log(x^2)^{1/2}|k), \qquad \alpha = 1, \ldots, 4, \\ \phi_5 &= k\,\text{sn}(\log(x^2)^{1/2}|k). \end{aligned} \tag{3.3}$$

Needless to say that in the limit $k \to 0(1)$ these solutions reduce to the meron (instanton) configuration (de Alfaro *et al* 1978). For $0 < k < 1$ the general solutions oscillate around the meron solution. These general solutions can be shown to be invariant under the O(4) group generated by complete (space plus internal) four-dimensional rotations. Note that in this model the meron and instanton solutions are invariant under $O(4) \otimes O(2)$ and O(5) groups respectively. Further, the energy–momentum tensor for these solutions is simply proportional to $\theta_{\mu\nu}^{\text{meron}}$, which is given by de Alfaro *et al* (1978). Finally, in this case too, even though the Euclidean energy and action are infinite, the corresponding Minkowski answers can be made finite by suitable conformal transformation.

It must be made clear here that the problem of ghost states is a serious drawback for these higher-derivative theories. Of course it is quite likely that, following Narnhofer and Thirring (1978), this problem may be tackled in the near future, but till then many workers will be sceptical of such theories. It may therefore be worthwhile to ask if one can obtain general classical solutions of a conventional four-dimensional σ-model. In the next section we show that it is possible provided one introduces self-interaction between the fields.

4. Four-dimensional σ-model with interaction

In this section we show that general classical solutions can be obtained of the four-dimensional O(4) Lagrangian (with Euclidean metric ++++)

$$\mathscr{L}=\tfrac{1}{2}\partial_\mu\phi_\alpha\,\partial_\mu\phi_\alpha-\tfrac{1}{4}\lambda\,(\phi_\alpha\phi_\alpha)^2,\qquad \alpha=1,\ldots,4. \tag{4.1}$$

A few comments are in order here. Firstly, notice that, even though ϕ is not dimensionless, the coupling constant λ is still dimensionless, so that the theory is conformal invariant. Secondly, for $\lambda>0$ this theory is asymptotically free. One may of course object to $\lambda>0$, since for $\lambda>0$ the theory is inconsistent with positivity (Coleman and Weinberg 1973), but then, as has been shown by Brandt (1976), the effective potential essentially vanishes in the exact theory due to asymptotic freedom. Thus it is quite possible that this theory may get around the positivity difficulty. It may therefore be worthwhile to discuss classical solutions of this model.

The equations of motion are

$$\Box\phi_\alpha-\lambda\phi^2\phi_\alpha=0. \tag{4.2}$$

Since ϕ_α has dimension one, we start with the ansatz

$$\phi_\alpha=(2/\lambda^{1/2})(x_\mu/x^2)f(x^2)\delta_{\mu\alpha}. \tag{4.3}$$

Substituting ansatz (4.3) into (4.2), it can be shown that f must satisfy the differential equation

$$x^4f''+x^2f'-f(1-f^2)=0, \tag{4.4}$$

where $f'\equiv \mathrm{d}f(x^2)/\mathrm{d}x^2$. The solution of this equation is well known to be (Murphy 1960)

$$f(x^2)=\left(\frac{2}{2-k}\right)^{1/2}\mathrm{dn}\left(\frac{\log x^2}{(2-k)^{1/2}}\,\middle|\,k\right), \tag{4.5}$$

where $0\leqslant k\leqslant 1$. We will show below that in the limit $k\to 0(1)$ these solutions go over to the meron (instanton) configuration. In general for $0<k<1$ these general solutions oscillate around the meron configuration.

Instanton. In the limit $k\to 1$

$$f(x^2)=\sqrt{2}\,\mathrm{sech}(\log x^2)=2\sqrt{2}\,x^2/(1+x^4), \tag{4.6}$$

so that

$$\phi_\alpha=[4\sqrt{2}\,x_\mu/\lambda^{1/2}(1+x^4)]\delta_{\mu\alpha},\qquad \alpha=1,\ldots,4. \tag{4.7}$$

For the instanton solution (4.7) the Lagrangian density (4.1) takes the form

$$\mathscr{L}=64(1-3x^4)(1-x^4)/\lambda\,(1+x^4)^4 \tag{4.8}$$

and the corresponding (Euclidean) action is finite,

$$A=\int\mathscr{L}\,\mathrm{d}^4x=64\pi^2/3\lambda. \tag{4.9}$$

Meron. In the limit $k \to 0$, $f(x^2) \to 1$, so that

$$\phi_\alpha = (2/\lambda^{1/2})(x_\mu/x^2)\delta_{\mu\alpha}, \tag{4.10}$$

which is the meron solution.

For this solution the Lagrangian density can be shown to be

$$\mathscr{L} = 4/\lambda x^4, \tag{4.11}$$

so that the Euclidean action is not well defined. However, following the treatment of de Alfaro *et al* (1978) this Lagrangian can be improved by means of a conformal transformation (combine translation, inversion, translation) to give

$$\mathscr{L} = 64a^4/\lambda(x-a)^4(x+a)^4. \tag{4.12}$$

We can now go to the physical Minkowski space by writing $x_4 = \mathrm{i}x_0$, and orient the vector a_μ along the time direction, $a_\mu = (0, 0, 0, 1)$, to give a solution which is regular everywhere. The total (Minkowski) action A that follows from here is finite and given by

$$A = \int \mathscr{L}\, \mathrm{d}^4x = 8\pi^3/\lambda. \tag{4.13}$$

For the general solutions (4.5) the Lagrangian density (4.1) reduces to

$$\mathscr{L} = [16/(2-k)^2\lambda x^4]\{(2-k-\mathrm{dn}^2)\,\mathrm{dn}^2 + [(2-k)^{1/2}k\,\mathrm{dn} + k^2\,\mathrm{sn}\,\mathrm{cn}]\,\mathrm{sn}\,\mathrm{cn}\}, \tag{4.14}$$

where

$$\mathrm{dn} \equiv \mathrm{dn}\left(\frac{\log x^2}{(2-k)^{1/2}}\,\middle|\, k\right).$$

The remarks of the previous section about finite Minkowski action also apply here.

The improved (traceless, divergenceless) energy–momentum tensor for this model is given by (Callan *et al* 1970)

$$\theta_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - \delta_{\mu\nu}\mathscr{L} - \tfrac{1}{6}(\partial_\mu\,\partial_\nu - \Box\delta_{\mu\nu})\phi^2. \tag{4.15}$$

Using the general solution (4.5) it is not difficult to show that

$$\theta_{\mu\nu}(x\,|\,k) = [(1-k)/(1-k/2)^2]\theta_{\mu\nu}^{\mathrm{meron}}, \tag{4.16}$$

where

$$\theta_{\mu\nu}^{\mathrm{meron}} = (4/3\lambda x^6)(\delta_{\mu\nu}x^2 - 4x_\mu x_\nu). \tag{4.17}$$

As expected $\theta_{\mu\nu}^{\mathrm{instanton}} = 0$, and for $0<k<1$ the $\theta_{\mu\nu}$ is just a multiple of $\theta_{\mu\nu}^{\mathrm{meron}}$. Not surprisingly the Euclidean energy is infinite, but the Minkowski energy can be made finite and non-zero by applying a suitable conformal transformation. In particular we find that

$$E^{\mathrm{meron}} = 8\pi^2/3\lambda. \tag{4.18}$$

Following the discussion of the last two sections (see also t'Hooft 1976) it is clear that, whereas the instanton and meron configurations are invariant under O(5) and $\mathrm{O}(4) \otimes \mathrm{O}(2)$ groups respectively, the general configurations (4.5) are only invariant under the O(4) group generated by complete four-dimensional rotations.

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